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THE STACK OF GERBES IN A QUOTIENT STACK

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Abstract. For a DM stack $\mathcal{X}$, Chen, Marcus and Úlfarsson ([3]) constructed a stack $\mathcal{G}_{\mathcal{X}}$ of gerbes in $\mathcal{X}$ that plays a key role in their setting up the theory of very twisted stable maps to $\mathcal{X}$. This stack is realized as a rigidification of the stack $S_{\mathcal{X}}$ of subgroups of the inertia stack of $\mathcal{X}$. In this article, we show that when $\mathcal{X}$ is a quotient stack, the stacks $\mathcal{S}_{\mathcal{X}}$ and $\mathcal{G}_{\mathcal{X}}$ are also quotient stacks.

1. Backgrounds

Throughout we work over complex numbers, and all schemes are locally of finite presentation unless otherwise stated.

1.1. Quotient stacks and gerbes

Definition 1.1. Let X be a scheme over a scheme S and G be a flat group scheme over S acting on X. Let $[X/G]$ be a category fibered in groupoids over S defined as follows:

1. An object of $[X/G]$ over T is a pair (P, F), where $P \to T$ is a G-principal bundle and $F : P \to Z$ is a G-equivariant morphism

$$\begin{array}{ccc} P & \xrightarrow{F} & X \\ \downarrow & & \\ T. & & \end{array}$$

2. A morphism between objects (P, F) and (P', F') over T and T', respectively, is a pair (Φ, u) such that the square is Cartisian and

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$F' \circ \Phi = F$

$$\begin{tikzcd} & & X \\ P \arrow[r, "\Phi"] \arrow[d] \arrow[rru, bend left, "F"] & P' \arrow[d] \arrow[ru, bend right, "F'"'] & \\ T \arrow[r, "u"] & T'. & \end{tikzcd}$$

Then it is routine to check that the fibered category $[X/G]$ is a stack ([5], [6]). The stack $[X/G]$ is called a *quotient stack*. It is well-known that if G is a flat group scheme locally of finite presentation over S, then $[X/G]$ is an Artin stack locally of finite presentation over S([4]). When $X = S$, and G acts on X trivially, $[X/G]$ is the *classifying stack* BG of G-principal bundles.

Remark 1.2. Note that for an object $\xi = (P, F)$ of $[X/G]$ over T, the group $\mathcal{A}ut_T(\xi)$ of automorphisms of ξ in the stack $[X/G]$ need not be a subgroup of the group of automorphisms of the G-principal bundle P. For example, let $S = X = *$ be a point. Then $\mathcal{A}ut_S(\xi)$ in the stack $BG = [*/G]$ with $P = G \times S$ is equal to G, while the group of automorphisms of the principal G-bundle P is the centralizer of G.

Definition 1.3 (Gerbes). An *étale gerbe* over a scheme S is a DM stack $\mathcal{A}$ over S such that

1. there exists an étale covering $S_i \to S$ such that $\mathcal{A}(S_i)$ is nonempty,
2. given two objects a, b of $\mathcal{A}(T)$ for a S-scheme T, there exists an étale covering $T_i \to T$ such that a_{T_i} and b_{T_i} are isomorphic in $\mathcal{A}(T_i)$.

More generally, a DM stack $\mathcal{F}$ over stack $\mathcal{Y}$ is an étale gerbe (over $\mathcal{Y}$) if for any morphism $U \to \mathcal{Y}$ from a scheme U, the pullback $\mathcal{F}_U$ is an étale gerbe over U.

2. Constructing some stacks

Now we give definitions of the stacks $\mathcal{S}_{\mathcal{X}}$ and $\mathcal{G}_{\mathcal{X}}$ and list properties of these stacks given in ([3]).

Definition 2.1. To a DM-stack $\mathcal{X}$, we associate a category $\mathcal{S}_{\mathcal{X}}$ fibered on groupoids over the category $(Schs/\mathbb{C})$ of schemes over $\mathbb{C}$ defined as follows.

(a) For a scheme T, an object of $\mathcal{S}_{\mathcal{X}}$ over T is a pair $(\xi, \alpha : \mathscr{H} \hookrightarrow \mathcal{A}ut_T(\xi))$, where ξ is an object of $\mathcal{X}(T)$, α is an injective morphism of group schemes, and $\mathscr{H}$ is finite and étale over T.

(b) An arrow sitting over $T \to T'$ from $(\xi, \alpha) \in \mathcal{S}_{\mathscr{X}}(T)$ to $(\xi', \alpha') \in \mathcal{S}_{\mathscr{X}}(T')$ is a morphism $\varphi : \xi \to \xi'$ such that there is a morphism of group schemes $\mathscr{H} \to \mathscr{H}'$ making the following diagram Cartisian

$$\begin{tikzcd}
\mathscr{H} \arrow[r] \arrow[d, "\alpha"] & \mathscr{H}' \arrow[d, "\alpha'"] \\
\mathcal{A}ut_T(\xi) \arrow[r] \arrow[d] & \mathcal{A}ut_{T'}(\xi') \arrow[d] \\
T \arrow[r] & T',
\end{tikzcd}$$

where $\mathcal{A}ut_T(\xi) \longrightarrow \mathcal{A}ut_{T'}(\xi')$ is induced by the morphism $\varphi : \xi \to \xi'$.

Remark 2.2. This definition is analogous to Definition 3.1.1 of [1] with the following differences. In our case, the subgroup $\mathscr{H}$ varies, and for an object $(\xi, \alpha : \mathscr{H} \hookrightarrow \mathcal{A}ut_T(\xi))$ of $S_{\mathcal{X}}$, the automorphism $\mathcal{A}ut_T(\xi, \alpha)$ is the normalizer of $\alpha(\mathscr{H})$ in $\mathcal{A}ut_T(\xi)$, while, in the latter case, $\mathscr{H} = T \times \mu_r$ is a fixed (trivial) group scheme, where μ_r is a group of r-th roots of unity, and $\mathcal{A}ut_T(\xi, \alpha)$ is the centralizer of $\alpha(\mathscr{H})$ in $\mathcal{A}ut_T(\xi)$.

For later use, we record the following result, due to Chen, Marcus and Úlfarsson ([3]).

Proposition 2.3 ([3]). *The followings hold for the stack $\mathcal{S}_{\mathcal{X}}$.*

1. *The fibered category $\mathcal{S}_{\mathcal{X}}$ is a DM-stack and the natural functor $\mathcal{S}_{\mathcal{X}} \to \mathcal{X}$ is representable and finite.*
2. *if $\mathcal{X}$ is a smooth DM-stack, then so is $\mathcal{S}_{\mathscr{X}}$.*
3. *If $\mathcal{X}$ is a proper DM-stack, then so is $\mathcal{S}_{\mathscr{X}}$.*

Proof. See Propositions 2.1.3 and 2.1.4 of [3]. □

2.1. Alternative definition of $\mathcal{S}_{\mathcal{X}}$

Definition 2.4. For a DM-stack $\mathcal{X}$, we define a 2-category $\mathcal{S}'_{\mathcal{X}}$, fibered over the category of schemes, as follows.

(a) For a scheme T, an objects of $\mathcal{S}'_{\mathcal{X}}$ is a triple $(\mathcal{A}, \tau, F)$, where $\mathcal{A}$ is an étale gerbe, τ is a section of the étale gerbe $\mathcal{A}$ over T, and F is a representable morphisms $\mathcal{A} \to \mathcal{X}$

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d] & \mathcal{X} \\ T. \arrow[u, bend left, "\tau"] & \end{tikzcd}$$

(b) An 1-arrow from (A, τ, F) to $(\mathcal{A}', \tau', F')$ over an arrow $u : T \to T'$ is a pair (Φ, η), where Φ is a morphism $\Phi : \mathcal{A} \to \mathcal{A}'$ making a cartesian square and $\Phi \circ \tau = \tau' \circ u$, and η is a natural transformation $\eta : F \Rightarrow F' \circ \Phi$

$$\begin{tikzcd} & & \mathcal{X} \\ \mathcal{A} \arrow[r, "\Phi"] \arrow[d] \arrow[urr, bend left, "F"] & \mathcal{A}' \arrow[d] \arrow[ur, bend right, "F'"'] & \\ T \arrow[r, "u"] & T'. & \end{tikzcd}$$

(c) A 2-arrow from (Φ, η) to (Φ_1, η_1) is a natural transformation $\sigma : \Phi \to \Phi_1$ inducing an equivalence, and compatible with η and η_1 in the sense that the following diagram is commutative;

$$\begin{tikzcd} & F \arrow[dl, "\eta"'] \arrow[dr, "\eta_1"] & \\ F' \circ \Phi \arrow[rr, "F(\sigma)"] & & F' \circ \Phi_1. \end{tikzcd}$$

Lemma 2.5 (Lemma 2.2.3 of [3]). *$\mathcal{S}'_{\mathcal{X}}$ is equivalent to a category.*

Definition 2.6. By abuse of notation, we use the same notation $\mathcal{S}_{\mathcal{X}}$ for the 1-category associated to the 2-category $\mathcal{S}_{\mathcal{X}}$.

Proposition 2.7. *There is an equivalence $\mathcal{S}'_{\mathcal{X}} \to \mathcal{S}_x$ of fibered categories.*

Proof. See the Definition 2.2.5 and Proposition 2.2.6. of [3]. $\square$

Remark 2.8. Notice that $\mathcal{S}_X$ parametrizes $(x, [H])$ for $x \in \mathcal{X}$ and equivalence classes $[H]$ with $H \subset \mathrm{A}ut(x)$, where $H \sim H'$ for $H, H' \in \mathcal{A}ut(x)$ if and only if $H = gHg^{-1}$ for some $g \in \mathcal{A}ut(x)$.

2.2. The stack of gerbes in $\mathcal{X}$

DEFINITION 2.9. For a DM-stack $\mathcal{X}$, we define a 2-category $\mathcal{G}_{\mathcal{X}}$, fibered over the category $(Schs/\mathbb{C})$, as follows.

(a) For a scheme T, an object of $\mathcal{G}_{\mathcal{X}}$ over T is a pair $(\mathcal{A}, F)$, where $\mathcal{A}$ is an étale gerbe over T and $F : \mathcal{A} \to \mathcal{X}$ is a representable morphism.

(b) An 1-arrow from (A, ϕ) to $(\mathcal{A}', \phi')$ over an arrow $u : T \to T'$ is a pair (Φ, η), where Φ is a morphism $\Phi : \mathcal{A} \to \mathcal{A}'$ making a cartesian square, and η is a natural transformation $\eta : F \Rightarrow F' \circ \Phi$.

$$\begin{tikzcd}
& & \mathcal{X} \\
\mathcal{A} \arrow[r, "\Phi"] \arrow[d] \arrow[urr, bend left, "F"] & \mathcal{A}' \arrow[d] \arrow[ur, bend right, "F'"'] & \\
T \arrow[r] & T'. &
\end{tikzcd}$$

(c) A 2-arrow from (Φ, η) to (Φ_1, η_1) is a natural transformation $\sigma : \Phi \to \Phi_1$ inducing an equivalence, and compatible with η and η_1 in the sense that the following diagram is commutative;

$$\begin{tikzcd}
& F \arrow[dl, "\eta"'] \arrow[dr, "\eta_1"] & \\
F' \circ \Phi \arrow[rr, "F(\sigma)"] & & F' \circ \Phi_1.
\end{tikzcd}$$

DEFINITION 2.10. As in the case of $\mathcal{S}'_{\mathcal{X}}$, the 2-category $\mathcal{G}_{\mathcal{X}}$ is equivalent to a category, which we denote by $\mathcal{G}_{\mathcal{X}}$ by abuse of notation.

2.3. Rigidification

For a DM-stack $\mathcal{Y}$ over S, $\mathcal{I}(\mathcal{Y})$ denotes the inertia stack of $\mathcal{Y}$. Let $\mathcal{H}$ be a subgroup stack of $\mathcal{I}(\mathcal{Y})$ (of finite type and étale over $\mathcal{Y}$). The rigidification of $\mathcal{Y}$ by $\mathcal{H}$ is a DM-stack over S, denoted $\mathcal{Y} /\!\!/ \mathcal{H}$, with a morphism $\rho : \mathcal{Y} \to \mathcal{Y} /\!\!/ \mathcal{H}$ satisfying the followings:

1. $\mathcal{Y}$ is an étale gerbe over $\mathcal{Y} /\!\!/ \mathcal{H}$.
2. For an object ζ of $\mathcal{Y}$ over T, the morphism of group schemes
$$\rho : \mathcal{A}ut_T(\zeta) \to \mathcal{A}ut_T(\rho(\zeta))$$
is surjective with kernel $\mathcal{H}_\zeta$.
3. If $\mathcal{H}$ is finite over $\mathcal{Y}$, then ρ is proper; while since $\mathcal{H}$ is étale, so is the morphism ρ.

See [2] for details on the notion of rigidification.

Now we take a rigidification of $\mathcal{S}_{\mathcal{X}}$. By Remark 2.2, for each object of $\mathcal{S}_{\mathcal{X}}(T)$

$$\xi_\alpha := (\xi, \alpha : \mathscr{H} \to \mathcal{A}ut_T(\xi)),$$

we have a normal embedding $\mathscr{H} \hookrightarrow \mathcal{A}ut_T(\xi_\alpha)$ of group schemes satisfying the compatibility condition for the arrow $T \to T'$. Let $\mathcal{H} \subseteq \mathcal{I}(\mathcal{S}_{\mathcal{X}})$ be the subgroup stack of $\mathcal{I}\mathcal{S}_{\mathcal{X}}$ defined by $\mathcal{H}_{\xi_\alpha} = \mathscr{H}$. Then $\mathcal{H}$ is finite and étale over $\mathcal{S}_{\mathcal{X}}$, and hence $\rho : \mathcal{S}_{\mathcal{X}} \to \mathcal{S}_{\mathcal{X}} /\!\!/ \mathcal{H}$ is proper and étale. By the following proposition, due to [3], $\mathcal{G}_{\mathcal{X}}$ can be viewed as a rigidification of $\mathcal{S}_{\mathcal{X}}$.

Proposition 2.11. *There is an equivalence $\mathcal{S}_{\mathcal{X}} /\!\!/ \mathcal{H} \to \mathcal{G}_{\mathcal{X}}$ of fibered categories.*

3. Main Theorem

For a reductive group G, let $FS(G)$ be the set of finite subgroups of G. Define an equivalence $\sim$ on $FS(G)$ as follows. $H \sim H'$ if $H' = gHg^{-1}$ for some $g \in G$. Let $\mathcal{I}(G)$ be the set of representatives from equivalence classes in $FS(G)/\sim$.

Proposition 3.1. *Let G be a reductive group acting on a scheme X, and assume that the quotient stack $\mathcal{X} := [X/G]$ is a DM-stack. Then there is an equivalence as fibered categories*

$$\mathscr{F} : \mathcal{S}_{\mathcal{X}} \to \coprod_{H \in \mathcal{I}(G)} [X^H/N_G(H)].$$

Proof. Define a functor $\mathscr{F} : S_{\mathcal{X}} \to \coprod_{H \in \mathcal{I}(G)} [X^H/N_G(H)]$ as follows. For a scheme T, let $(\xi = (P, F), \alpha : \mathcal{H} \hookrightarrow \mathcal{A}ut_T(\xi))$ be an object in $S_{\mathcal{X}}$ over T, where $P \to T$ is a G-principal bundle and $F : P \to X$ is a G-equivariant. By taking an étale cover of T, we may assume P is a trivial G-principal bundle over T, and so $\mathcal{A}ut_T(\xi)$ is a subgroup scheme of the trivial group scheme $G \times T$ and also $\mathcal{H} = T \times H$ for some $H \subset G$. Suppose WLOG that $H \in \mathcal{I}(G)$. Note that for objects in a connected component of $\mathcal{S}_{\mathcal{X}}$, the chosen H does not change. The embedding $\mathcal{H} \hookrightarrow \mathcal{A}ut_T(\xi)$ implies that whenever ρ is a section of $\mathcal{H} = T \times H$ over T, we have $F \circ \rho = F$. That is, writing $\rho = h(t) \in H$ $(t \in T)$, we have

$$F(h(t)p) = F(p)$$

for each $t \in T$ and $p \in P$. In particular, each constant section $h \in H$ preserves the fiber $f^{-1}(y)$ for each $y \in X^H$. Let $N = N_G(H)$ be the normalizer of H in G, and

$$P_H := \{p \in P \,|\, f(p) \in Y^H\}.$$

Then P_H is a principal N-bundle over T. To see this, note that $x \in X^H$ iff $gx \in X^{gHg^{-1}}$ for $g \in G$, and so $g \in N$ if and only if $V^H = V^{gHg^{-1}}$. From this it is immediate that P_H is a principal N-bundle and the G-equivariant morphism $F : P \to X$ restricts to P_H to give a N-equivariant morphism $F_H : P_H \to X^H$. Therefore we obtain an object of the quotient stack $[X^H/N]$ over T. Then we define $\mathscr{F}((\xi, \alpha)) := (P_H, F_H)$.

Now we construct a functor $\mathscr{F}' : \coprod_{H \in \mathcal{I}(G)} [X^H/N_G(H)] \to \mathcal{S}_{\mathcal{X}}$. Let $\varrho := (Q, f)$ be an object of $[X^H/N]$, i.e., Q is a principal N-bundle over T and f is a N-equivariant morphism to X^H. Let $P := G \times_N Q$, and define $\tilde{f} : P = G \times_N Q \to G \times_N X^H$ by $\tilde{f}(g, q) := (g, f(q))$ for $(g, q) \in G \times_N Q$. By abuse of notation, denote by $\tilde{f} : P \to X$ the composite of $\tilde{f}$ and the embedding $G \times_N X^H \hookrightarrow X$, $(g, x) \mapsto gx$. Then $\tilde{f} : P \to X$ is a G-equivariant morphism. Let ξ be an object in $\mathcal{X} = [X/G]$ corresponding to $\tilde{f} : P \to X$, i.e., $\xi := (P, \tilde{f})$. Then since the equivariant morphism $\tilde{f} : P \to X$ is a lift of $f : Q \to X^H$, étale locally on T the group scheme $\mathcal{A}ut_T(\xi)$ contains $\mathscr{H} := T \times H$. Namely we have the injective morphism $\mathscr{H} \overset{\alpha}{\hookrightarrow} \mathcal{A}ut_T(\xi)$. Put $\mathscr{F}'(\varrho) := (\xi, \alpha)$. On the other hand, as above, since $G \times_N P_H = P$, and $F : P \to X$ is completely determined by the restriction $F_H : P_H \to X^H$, we obtain the relations

$$\widetilde{(F_H)} = F, \;\; (\tilde{f})_H = f.$$

Therefore it is immediate that for $(\xi, \alpha) \in \mathcal{S}_{\mathcal{X}}$

$$\mathscr{F}' \circ \mathscr{F}((\xi, \alpha)) = (\xi, \alpha),$$

and for $\varrho \in \coprod_{H \in \mathcal{I}(G)} [X^H/N_G(H)]$

$$\mathscr{F} \circ \mathscr{F}'(\varrho) = \varrho.$$

Now we prove the fully faithfulness. For objects (ξ, α) and (ξ', α') over T and T', respectively, let us show that there is one to one correspondence

$$Mor((\xi, \alpha), (\xi', \alpha')) \overset{\sim}{=} Mor(\mathscr{F}(\xi, \alpha), \mathscr{F}(\xi', \alpha')).$$

Assume that $\mathscr{F}((\xi,\alpha))$ and $\mathscr{F}((\xi',\alpha'))$ are objects in $[X^H/N]$ for $H \in \mathcal{I}(G)$. Let ϑ be an element of $Mor((\xi,\alpha),(\xi',\alpha'))$. Then there are morphisms Φ and u, with $\vartheta = (\Phi, u)$, making the diagram below Cartisan

$$\begin{tikzcd} & & X \\ P \arrow[r, "\Phi"] \arrow[d] \arrow[urr, bend left, "F"] & P' \arrow[d] \arrow[ur, bend right, "F'"'] & \\ T \arrow[r, "u"] & T'. & \end{tikzcd}$$

Then image $\mathscr{F}(\vartheta)$ is simply a pair (Φ_H, u) in the diagram

$$\begin{tikzcd} & & X^H \\ P_H \arrow[r, "\Phi_H"] \arrow[d] \arrow[urr, bend left, "F_H"] & P'_H \arrow[d] \arrow[ur, bend right, "F'_H"'] & \\ T \arrow[r, "u"] & T', & \end{tikzcd}$$

where $\Phi_H : P_H \to P'_H$ is the restriction of $\Phi : P \to P'$.

Let $\eta = (\phi, u)$ be an element of $Mor(\mathscr{F}(\xi,\alpha), \mathscr{F}(\xi',\alpha'))$. Let $P = G \times_N Q$ and $P' = G \times_N Q'$. Then $\phi : Q \to Q'$ naturally extend to a morphism $\tilde{\phi} : P \to P'$. Furthermore, as above, we have the relations $\widetilde{(\Phi_H)} = \Phi$ and $(\tilde{\phi})_H = \phi$. This proves the fully faithfulness of $\mathscr{F}$. Therefore, $\mathscr{F}$ is an equivalence.

$\square$

Let us identify two stacks $\mathcal{S}_{\mathcal{X}} = \coprod_{H\in\mathcal{I}(G)}[X^H/N_G(H)]$. Then by Proposition 2.11, we have

Corollary 3.2. *The DM-stack $\mathcal{G}_{\mathcal{X}}$ is a quotient stack*

$$\mathcal{G}_{\mathcal{X}} = \coprod_{H\in\mathcal{I}(G)} [X^H/\widetilde{N}],$$

where $\widetilde{N} = N_G(H)/H$.

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